

# General properties and some solutions of generalized Einstein - Eddington affine gravity I

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## Abstract

After a brief exposition of the simplest class of *affine theories of gravity* in multidimensional space-times with *symmetric connections*, we consider the spherical and cylindrical reductions of these theories to two-dimensional *dilaton-vecton gravity* (DVG) field theories. The distinctive feature of these theories is the presence of a massive/tachyonic vector field (*vecton*) with essentially nonlinear coupling to the dilaton gravity. In the massless limit, the classical DVG theory can be exactly solved for a rather general coupling depending only on the field tensor and the dilaton. We show that the vecton field can be consistently replaced by a new effectively massive scalar field (*scalaron*) with an unusual coupling to dilaton gravity (DSG).

Then we concentrate on considering the DVG models derived by reductions of  $D = 3$  and  $D = 4$  affine theories. In particular, we introduce the most general cylindrical reductions that are often ignored. The main subject of our study is the static solutions with horizons. We formulate the general conditions for the existence of the regular horizons and find the solutions of the static DVG/DSG near the horizons in the form of locally convergent power - series expansion. For an arbitrary regular horizon, we find a local generalization of the Szekeres - Kruskal coordinates. Finally, we consider one-dimensional integrable and nonintegrable DSG theories with one scalar. We analyze simplest models having three or two integrals of motion, respectively, and introduce the idea of a *topological portrait* giving a unified qualitative description of static and cosmological solutions of some simple DSG models.

## 1 Introduction

The present observational data strongly suggest that Einstein's gravity must be modified, one of the popular modifications being provided by superstring ideas. In view of the mathematical problems of the string theory, other, much simpler, modifications of gravity that affect only the gravitational sector (not touching other interactions) are also popular. One such modification was proposed and studied in [1] - [4].

It is based on Einstein's idea (1923)<sup>1</sup> to formulate the gravity theory in a non - Riemannian space with a symmetric connection by use of a special variational principle that allows one to determine the connection from a 'geometric' Lagrangian. This Lagrangian is assumed to depend of the generalized Ricci curvature tensor and of other fundamental tensors and is varied in the connection coefficients. A new interpretation and a generalization of this approach was developed in [2] - [4] for an arbitrary space - time dimension  $D$ . In general, the resulting theory supplements the standard general relativity with dark energy (the cosmological constant, in the first approximation), neutral massive (or tachyonic) vector field (*vecton*) and (after a dimensional reduction to  $D = 4$ ) with  $(D - 4)$  massive (or tachyonic) scalar fields.

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<sup>1</sup>The references to his papers as well as to related work of Weyl and Eddington can be found in [1] - [4].

The further details of the theory depend on a concrete choice of the geometric Lagrangian, and then the corresponding physical theory can be described by the effective Lagrangian depending on the metric, vecton and scalar fields. In the geometric theory, there are no dimensional constants, while any fundamental tensor has dimension of some power of length (assuming  $c = 1$ ). We proposed a class of the geometric Lagrangian densities depending on the symmetric,  $s_{ij}$ , and antisymmetric,  $a_{ij}$ , parts of the generalized Ricci tensor as well as of a fundamental vector obtained by contracting the connection. Requiring the geometric ‘action’ (i.e. the integral of the geometric Lagrangian density) to be dimensionless we can enumerate all possible actions. Thus, for  $D = 2n$  and  $D = 2n + 1$  we can construct  $n + 1$  independent scalar densities from the tensors  $s_{ij}$  and  $a_{ij}$  having the weight two and dimension  $L^{-D}$ . Each Lagrangian is the square root of an arbitrary linear combination of these densities (of course, one can also take a linear combination of the square roots).

The simplest useful density of this sort in any dimension is the square root of  $\det(s_{ij} + \bar{\lambda}a_{ij})$ , where  $\bar{\lambda}$  is a number.<sup>2</sup> The effective physical Lagrangian is the sum of the standard Einstein term, the vecton mass term, and the term proportional to  $\det(g_{ij} + \lambda f_{ij})$  to the power  $\nu \equiv 1/(D - 2)$ , where  $g_{ij}$  and  $f_{ij}$  are the metric and the vecton field tensors (conjugate to  $s_{ij}$  and  $a_{ij}$ ),  $\lambda$  is the number related to  $\bar{\lambda}$ . The last term has the dimensional multiplier, which in the limit of small field  $f_{ij}$  produces the cosmological constant. For  $D = 4$  we therefore have the term first introduced by Einstein but now usually called the Born-Infeld or brane Lagrangian. For  $D = 3$  we have the Einstein - Proca theory, which is very interesting for studies of nontrivial space topologies. We will derive and study several analytical solutions of the one-dimensional reductions of this theory but their topological and physical meaning will be discussed elsewhere.

A simplest dimensional reduction from  $D > 4$  to  $D = 4$  produces  $(D - 4)$  scalar fields which are also geometrically massive (or, tachyonic). The complete theory is very complex, even at the classical level. Its spherically symmetric sector is described by a much simpler 1+1 dimensional dilaton gravity coupled to one massive vector and to several scalar fields. This dilaton gravity coupled to the vecton and massive scalars as well as its further reductions to 1+0/0+1 dimensional ‘cosmological’ and ‘static’ theories were first formulated in [1] - [4], and here we begin systematic studies of their general properties and solutions. These studies are somewhat simplified by transforming the vecton field into a new massive scalar field, which is possible (on the mass shell in the 1+1 dimensional reduction).<sup>3</sup>

A class of exact solutions of the 1+1 dimensional dilaton-vecton gravity theory can be derived in the zero - mass case. We have shown that, if the mass of the vector field is zero and the scalar fields vanish, the dilaton-vecton gravity is explicitly integrable. This is true for any  $D$ , but the more or less explicit solutions are possible only for  $D = 3, 4, 5$  (with arbitrary other parameters). One may hope that our solution can be used as a first approximation in a perturbative expansion and thus our approach opens a way to use a perturbation theory in the mass parameter. However, as can be seen from our treatment of the exact massive equations, such a perturbation must be rather nontrivial. In fact, as is demonstrated below, the 1+1 dimensional DVG, with a massive vector field, can be transformed into a dual DSG, with a scalar field replacing the vecton.<sup>4</sup> This transformation allows us to use all general results obtained in recent years for the dilaton gravity coupled to the standard scalar fields. However, it shows that the perturbation theory may be ‘singular’ because the equations of motion become degenerate in the limit of vanishing mass

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<sup>2</sup>Einstein used as the Lagrangian Eddington’s scalar density  $\sqrt{|\det r_{ij}|}$ , where  $r_{ij} \equiv s_{ij} + a_{ij}$ .

<sup>3</sup>In addition to some formal simplifications, the transformation may shed a new light on some theoretical problems, especially, in cosmology. Indeed, new cosmological theories operate with several exotic scalar fields (inflaton, phantoms, tachyons), which are usually introduced *ad hoc*. In the models considered here, such particles may appear in a consistent theoretical framework.

<sup>4</sup>As distinct from the normal scalar fields, the scalaron has a different coupling to gravity and may have abnormal signs of the kinetic term (phantom) or mass term (tachyon).

parameter (a well - known and more difficult problem of this sort is the so called boundary layer problem in hydrodynamics).

After this brief overview of the results obtained in [1] - [4] and some ideas of the present study, we give main equations which will be used below. Here we consider only the simplest **geometric Lagrangian**,

$$\mathcal{L}_g = \sqrt{-\det(s_{ij} + \bar{\lambda}a_{ij})}, \quad (1)$$

where the minus sign is taken because  $\det(s_{ij}) < 0$  (due to the local Lorentz invariance) and we naturally assume that the same is true for  $\det(s_{ij} + \bar{\lambda}a_{ij})$  (to reproduce Einstein's general relativity with the cosmological constant in the limit  $\bar{\lambda} \rightarrow 0$ ).

Following the steps of Ref. [2], we may write the corresponding **physical Lagrangian**

$$\mathcal{L}_{ph} = \sqrt{-g} \left[ -2\Lambda [\det(\delta_i^j + \lambda f_i^j)]^\nu + R(g) + m^2 g^{ij} a_i a_j \right], \quad \nu \equiv 1/(D-2), \quad (2)$$

which should be varied with respect to the metric and the vector field;  $m^2$  is a parameter depending on the chosen model for affine geometry and on  $D$  (see [1] - [4]). This parameter can be positive or negative. In Einstein's first model it is negative. When the vecton field is zero, we have the standard Einstein gravity with the cosmological constant. Making the dimensional reduction from  $D \geq 5$  to  $D = 4$ , we obtain the Lagrangian describing the vecton  $a_i$ ,  $f_{ij} \sim \partial_i a_j - \partial_j a_i$  and  $(D-4)$  scalar fields  $a_k$ ,  $k = 4, \dots, D$ .

It is interesting to note that, for  $D = 3$ , the Lagrangian (2) is bilinear in the vecton field, and in the approximation  $a_i = 0$  it gives the three-dimensional gravity with the cosmological constant. The three-dimensional gravity was studied by many authors<sup>5</sup> and this may significantly simplify the study of solutions of the new theory. Unfortunately, we could not find works including into consideration both the cosmological constant and massive/tachyonic vectons and thus we begin our study by considering basic features of the simplest solutions of the theory not touching most interesting topological and quantum aspects.

We first consider the simplest spherical dimensional reduction from the  $D$ -dimensional theory (2) to the two-dimensional dilaton-vecton gravity (DVG) and then further reduce it to one-dimensional static or cosmological-type theories. The next step is considering cylindrical reductions of the  $D$ -dimensional theory to more complex two-dimensional dilaton gravity theories coupled to several scalar fields. For simplicity, we consider this dimensional reduction only in the dimensions four and three. As is well known, in the  $D = 4$  case, in addition to the dilaton, there appear 'geometric'  $\sigma$ -model scalar fields and vector pure gauge fields that produce an effective potential introduced in [21]. These two-dimensional scalars look like scalar matter fields obtained by dimensional reductions from higher dimensional supergravity theory. Like those fields, they are classically massless. However, in general, the  $\sigma$ -model coupling of the scalar fields is supplemented by the mentioned effective potential depending on the scalar fields and on the 'charges' of the 'geometric' pure gauge fields. This theory is very complex and not integrable even after reduction to dimensions 1+0 or 0+1 (see [21]). Of course, adding the vecton coupling makes the theory much more complex and in this paper we consider only the simplest variants.

Much simpler, although quite nontrivial, is the three-dimensional theory (2) (with  $D = 3$ ). We hope that its dimensional reductions to two-dimensional and one-dimensional DVG theories may give some insight into properties of the more realistic four-dimensional theory and thus study these models in more detail. For example, the cylindrical reductions are of great interest

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<sup>5</sup>See, e.g., very interesting papers of the last century, [5] - [14]. Many ideas and results of these and other studies of the 3-gravity before 1997 were summarized in a beautiful review [15] (see also [16]). More recent studies of the three-dimensional gravity reveal new aspects of its relation to string theory and modern ideas on quantizing gravity, see, e.g. [17] - [20].

in connection with the theory of nonlinear gravitational waves<sup>6</sup>.

In this paper we mainly discuss low-dimensional theories ( $D = 1, 2, 3, 4$ ) that allow one to consistently treat some solutions of realistic higher dimensional theories. We mostly concentrate on mathematical problems and do not discuss in any detail dimensions and values of the physical parameters as well as physics meaning of the obtained solutions. Note also that in the context of modern ideas on inflation, multiverse, etc., the main parameters of a fundamental theory of gravity cannot be theoretically determined. In particular, we do not know the sign and the magnitude of the cosmological constant (or any other parameter giving a fundamental scale of length). Even the dimension of the space - time (and, possibly, its signature) should be considered as a free parameter that can be estimated in the context of a concrete scenario of the multiverse evolution or by anthropic considerations. In practice, this means that we should regard theories with any parameters, in any space-time dimension as equally interesting, at least, theoretically.

## 2 Dimensional reductions of the generalized gravity theory

Let us outline the main reductions of the model (2) in the dimensions  $D = 3, 4$ . Due to natural space and time restrictions we give only an overview of our results<sup>7</sup>. First, consider a rather general Lagrangian in the  $D$ -dimensional **spherically symmetric** case:

$$ds_D^2 = ds_2^2 + ds_{D-2}^2 = g_{ij} dx^i dx^j + \varphi^{2\nu} d\Omega_{D-2}^2(k), \quad (3)$$

where  $\nu \equiv (D - 2)^{-1}$  and  $k = 0, \pm 1$ . The standard spherical reduction of (2) gives the effective Lagrangian

$$\mathcal{L}_D^{(2)} = \sqrt{-g} \left[ \varphi R(g) + k_\nu \varphi^{1-2\nu} + \frac{1-\nu}{\varphi} (\nabla \varphi)^2 + X(\varphi, \mathbf{f}^2) - m^2 \varphi \mathbf{a}^2 \right], \quad (4)$$

where  $a_i(t, r)$  has only two nonvanishing components  $a_0, a_1$ ,  $f_{ij}$  has just one independent component  $f_{01} = a_{0,1} - a_{1,0}$ ; the other notations are:  $\mathbf{a}^2 \equiv a_i a^i \equiv g^{ij} a_i a_j$ ,  $\mathbf{f}^2 \equiv f_{ij} f^{ij}$ ,  $k_\nu \equiv k(D - 2)(D - 3)$ , and, finally,<sup>8</sup>

$$X(\varphi, \mathbf{f}^2) \equiv -2\Lambda \varphi \left[ 1 + \frac{1}{2} \lambda^2 \mathbf{f}^2 \right]^\nu, \quad (5)$$

(here, the parameter  $\lambda$  is related to  $\lambda$  in (2) but does not coincide with it in general).

Sometimes, it is convenient to transform away the dilaton kinetic term by using the Weyl transformation, which in our case is the following:

$$g_{ij} = \hat{g}_{ij} w^{-1}(\varphi), \quad w(\varphi) = \varphi^{1-\nu}, \quad \mathbf{f}^2 = w^2 \hat{\mathbf{f}}^2, \quad \mathbf{a}^2 = w \hat{\mathbf{a}}^2. \quad (6)$$

Applying this transformation to (4) and omitting the hats we find the transformed Lagrangian

$$\mathcal{L}_{DW}^{(2)} = \sqrt{-g} \left[ \varphi R(g) + k_\nu \varphi^{-\nu} - 2\Lambda \varphi^\nu \left( 1 + \frac{1}{2} \lambda^2 \varphi^{2(1-\nu)} \mathbf{f}^2 \right)^\nu - m^2 \varphi \mathbf{a}^2 \right]. \quad (7)$$

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<sup>6</sup>The first static cylindrical reductions were obtained in [22] - [23] and the first cylindrical nonlinear wave solutions were derived in [24]. Cylindrical solutions may be of interest for the theory of cosmic strings. They can also be related to static axially symmetric solutions like those derived in [25]. The cylindrical solutions in the presence of a massive vector field were not discussed in literature, to the best of our knowledge.

<sup>7</sup>For a more rigorous discussion and applications of the reductions in  $D = 3, 4$  see [26] - [29]

<sup>8</sup>The expression for the determinant in Eq.(2) in terms of  $f_{ij}$  for  $D = 4$  written in [2] contains also the fourth-order term the vector field. It is not difficult to see that in both spherical and cylindrical reductions (see the end of this Section) this term vanishes and thus Eq.(5) is valid. In fact, this is also true in any dimension  $D$ .

When  $D = 3$  we have  $\nu = 1$ ,  $k_\nu = 0$ , Weyl's transformation is trivial and the Lagrangian is

$$\mathcal{L}_3^{(2)} = \sqrt{-g} \left[ \varphi R(g) - 2\Lambda\varphi - \lambda^2 \Lambda \varphi \mathbf{f}^2 - m^2 \varphi \mathbf{a}^2 \right]. \quad (8)$$

These two-dimensional theories are essentially simpler than their parent higher dimensional theories. In particular, we show below that in dimension two the massive vector field theory can be transformed into a scalar dilaton gravity (DSG) model which is easier to analyze. Unfortunately, these DSG models and their further reductions to dimension one (static and cosmological reductions) are also essentially non-integrable. It is well known that the massless case, being a pure dilaton gravity, is classically integrable even for an arbitrary coupling of the massless vecton to gravity (see, e.g., [30] - [33] and reference therein, especially, in the review [33]). Having this in mind we try to find some additional integrals of motion.

The next simplified theory is obtained in a **cylindrically symmetric** case. We consider here only  $D = 3$  and  $D = 4$  cases. The general cylindrical reduction was discussed in detail in [21] and here we only summarize the main results. The most general cylindrical Lagrangian can be derived by applying the general Kaluza reduction to  $D = 4$ . The corresponding metric may be written as

$$ds_4^2 = (g_{ij} + \varphi \sigma_{mn} \varphi_i^m \varphi_j^n) dx^i dx^j + 2\varphi_{im} dx^i dy^m + \varphi \sigma_{mn} dy^m dy^n, \quad (9)$$

where  $i, j = 0, 1$ ,  $m, n = 2, 3$ , all the metric coefficients depend only on the  $x$ -coordinates  $(t, r)$ , and  $y^m = (\phi, z)$  are coordinates on the two-dimensional cylinder (torus). Note that  $\varphi$  plays the role of a dilaton and  $\sigma_{mn}$  ( $\det \sigma_{mn} = 1$ ) is the so - called  $\sigma$ -field. The reduction of the Einstein part of the four-dimensional Lagrangian,  $\sqrt{-g_4} R_4$ , can be written as:

$$\mathcal{L}_{4C}^{(2)} = \sqrt{-g} \left[ \varphi R(g) + \frac{1}{2\varphi} (\nabla \varphi)^2 - \frac{\varphi}{4} \text{tr}(\nabla \sigma \sigma^{-1} \nabla \sigma \sigma^{-1}) - \frac{\varphi^2}{4} \sigma_{mn} \varphi_i^m \varphi^{nij} \right], \quad (10)$$

where  $\varphi_{ij}^m \equiv \partial_i \varphi_j^m - \partial_j \varphi_i^m$ . These Abelian gauge fields  $\varphi_i^m$  are not propagating and their contribution is usually neglected. We proposed in [21] to take them into account by solving their equations of motion and writing the corresponding effective potential (similarly to what we are doing below in the spherically symmetric vecton gravity). Introducing a very convenient parametrization of the matrix  $\sigma_{mn}$ ,

$$\sigma_{22} = e^\eta \cosh \xi, \quad \sigma_{33} = e^{-\eta} \cosh \xi, \quad \sigma_{23} = \sigma_{32} = \sinh \xi, \quad (11)$$

one can exclude the gauge fields  $\varphi_i^m$  and derive the effective action

$$\mathcal{L}_{\text{eff}}^{(2)} = \sqrt{-g} \left[ \varphi R(g) + \frac{1}{2\varphi} (\nabla \varphi)^2 + V_{\text{eff}}(\varphi, \xi, \eta) - \frac{\varphi}{2} [(\nabla \xi)^2 + (\cosh \xi)^2 (\nabla \eta)^2] \right]. \quad (12)$$

where the effective geometric potential,

$$V_{\text{eff}}(\varphi, \xi, \eta) = -\frac{\cosh \xi}{2\varphi^2} \left[ Q_1^2 e^{-\eta} - 2Q_1 Q_2 \tanh \xi + Q_2^2 e^\eta \right], \quad (13)$$

depends on two arbitrary real constants  $Q_m$ , which may be called ‘charges’ of the Abelian geometric gauge fields  $\varphi_{ij}^m$ .

This representation of the action is more convenient for writing the equations of motion, for further reductions to dimensions  $(1+0)$ , and  $(0+1)$  as well as for analyzing special cases, such as  $Q_1 Q_2 = 0$ ,  $\xi \eta \equiv 0$ .<sup>9</sup> The static solutions of the theory (13) with  $Q_1 Q_2 \neq 0$  have horizons while the exact solutions derived in [21] for  $Q_1 = Q_2 = 0$  and nonvanishing  $\sigma$ -fields  $\xi, \eta$  have no horizons at all, in accordance with the general theorem of papers [31]. An interesting special case

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<sup>9</sup>It is also closer to the original Einstein - Rosen equations for nonlinear gravitational waves [24], which can be obtained by putting  $Q_1 = Q_2 = 0$  and  $\xi \equiv 0$ . When  $Q_1 Q_2 \neq 0$ ,  $\xi$  and  $\eta$  cannot be identically zero.

can be obtained if we choose  $Q_1 = 0$ ,  $Q_2 \neq 0$ ,  $\xi \equiv 0$ . In the Weyl frame the effective Lagrangian can be written in the form

$$\mathcal{L}_W^{(2)} = \sqrt{-g} \left[ \varphi R(g) - \frac{Q_1^2}{2\varphi^{5/2}} e^{-\eta} - \frac{\varphi}{2} (\nabla\eta)^2 \right]. \quad (14)$$

This is a standard dilaton gravity coupled to the scalar field  $\eta$ , with the potential depending both on the scalar field and dilaton. If  $Q_1 \neq 0$ , there exists a static solution with a horizon, which disappears in when  $Q_1$  vanishes. Of course, the horizon exists also in pure dilaton gravity, when  $\eta \equiv 0$ . In [31] we studied in some detail the models with the potentials independent of the scalar. Below we show that some results can be derived in some more general models with ‘separable’ potentials  $V(\phi, \psi) = v_1(\phi)v_2(\eta)$ . In particular, we show that one of the integrals of motion in the dilaton gravity coupled to massless scalars derived in [31] may exist also in some models with separable potentials that are of interest in the context of the present study.

We see that the general cylindrical action is a very complex two-dimensional theory and even its one-dimensional reductions are rather complex and in general not integrable. Of course, adding the vecton sector does not make it simpler and more tractable. It deserves further studies mainly because it is much more realistic (from the physics point of view) than the spherically symmetric theory and still simpler than the axially symmetric theory. Also, it is of physics interest because it can describe cosmic strings and in this direction one may hope to find some effects produced by the vecton coupling to gravity, i.e. traces of the affine geometry.

Much more tractable is the *three-dimensional* cylindrical space-time. The metric can be obtained by the obvious reduction of (9),

$$ds_3^2 = (g_{ij} + \varphi_i \varphi_j) dx^i dx^j + 2\varphi_i dx^i dy + \varphi dy^2, \quad (15)$$

and the corresponding Einstein Lagrangian is simply

$$\mathcal{L}_{3c}^{(2)} = \sqrt{-g} \varphi \{ R(g) - \frac{\varphi}{4} \varphi_{ij} \varphi^{ij} \}, \quad \varphi_{ij} \equiv \varphi_{i,j} - \varphi_{j,i}. \quad (16)$$

Using the equation of motion for  $\varphi_{ij}$  and introducing the corresponding effective potential (see [21] and more general derivation below) we derive the following two-dimensional dilaton gravity

$$\mathcal{L}_{\text{eff}}^{(2)} = \sqrt{-g} \{ \phi R(g) - 8Q^2 \phi^{-3} \}, \quad \phi \equiv \sqrt{\varphi}. \quad (17)$$

As distinct from the cylindrical reduction of the four-dimensional pure Einstein theory (corresponding to  $Q = 0$  in (17)), this theory has a horizon (see Section 4).

It is not difficult to add the vecton part to the cylindrically symmetric Lagrangians. In fact the terms  $-m^2 \varphi \mathbf{a}^2$ ,  $X(\varphi, \mathbf{f}^2)$  (see (5)) are invariant and have the same form in any dimension. Therefore we can simply add the expressions (12), (16) to the gravitational Lagrangians also in the cylindrical case. However, there exist different cylindrical reductions of the vecton potential  $a_i$ . For example, unlike the spherical case, these fields may be nonzero for  $i = 0, 1, 2$  and correspondingly  $e_1 \equiv f_{01} \equiv \partial_0 a_1 - \partial_1 a_0 \neq 0$ ,  $e_2 \equiv f_{02} \equiv \partial_0 a_2 \neq 0$ ,  $h_3 \equiv f_{12} \equiv \partial_1 a_2 \neq 0$ .<sup>10</sup> We see that the component  $a_2 \equiv a_\varphi$  of the vector field  $a_i$  behaves like an additional scalar field and, in addition to the two-dimensional vector field  $(a_0, a_1)$  we have up to three scalar matter fields  $a_2, \xi, \eta$  with a rather complex interaction to the dilaton gravity. This means that analyzing cylindrical solutions is more difficult than that of the spherical ones (see, e.g., [28] for different exact cylindrical solutions of Einstein - Maxwell theory). Our consideration of the cylindrical vecton solutions will be by necessity only fragmentary and superficial. In particular, in next Section we consider only the simple case of two potentials  $(a_0, a_1)$  and one field  $f_{01}$ .

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<sup>10</sup>Here  $e_i \equiv a_{0i}$ ,  $h_i \equiv \varepsilon_{ijk} a_{jk}$ . In a diagonal metric  $g_{ij} = g_i \delta_{ij}$  the fourth-order term in the determinant in Eq.(2) is proportional to  $(e_i h_i)^2 \exp \sum 2g_k$  and is seen to vanish. This argument obviously works in any dimension.

### 3 Nonlinear coupling of gauge fields and vector - scalar duality

In the dimension  $D = 2$  all fields (vector, spinor, ...) are practically equivalent to scalar ones. Such equivalence is widely known for massless Abelian gauge fields (see, e.g., [34], [35] and references therein). There exist also examples of such equivalence for some non - Abelian theories (see, e.g., [36]). The aim of this Section is more modest – to establish a standard map of (massive) Abelian vector fields to scalar fields. We do not attempt to find the most general result in this direction and restrict our consideration to a rather general class of DVG coupled to some scalar ‘matter’ fields. This class includes all theories of the previous sections.

Suppose that in place of the standard Abelian gauge field term,  $X(\varphi, \psi) \mathbf{f}^2$ , the Lagrangian contains a more general coupling of the gauge field  $f_{ij} = \partial_i a_j - \partial_j a_i$  to dilaton and scalar fields,  $X(\varphi, \psi; \mathbf{f}^2)$ , where  $\mathbf{f}^2 \equiv f_{ij} f^{ij}$ . Using the Weyl transformation (if necessary) we may write a fairly general two-dimensional Lagrangian as

$$\mathcal{L}^{(2)} = \sqrt{-g} \left[ \varphi R + V(\varphi, \psi) + X(\varphi, \psi; \mathbf{f}^2) + Z(\varphi) \mathbf{a}^2 + \sum Z(\varphi, \psi) (\nabla \psi)^2 \right]. \quad (18)$$

We need not specify the number of the scalar matter fields and therefore omit the summation indices for the matter fields  $\psi$  and their  $Z$ -functions. For the fields, having positive kinetic energy, all the  $Z$ -functions in (18) are negative and usually proportional to  $\varphi$ . We may call them ‘normal matter’ fields or simply normal fields. This is not true for the dilaton  $\varphi$  as seen in Eq.4; in particular the sign if the dilaton term can be changed and it can be even transformed to zero.

In [34] we considered the massless vector fields ( $Z \equiv 0$ ) and proposed to use instead of (18) the effective Lagrangian not containing the Abelian gauge fields (for a detailed proof see [35]):

$$\mathcal{L}_{\text{eff}}^{(2)} = \sqrt{-g} \left[ \varphi R + V(\varphi, \psi) + X_{\text{eff}}(\varphi, \psi; q) + \sum Z(\varphi, \psi) (\nabla \psi)^2 \right]. \quad (19)$$

Here  $q$  are integration constants (charges) defined by the solution of the equations for  $f^{ij}$ ,

$$2\partial_j (\sqrt{-g} X' f^{ij}) = Z(\varphi) \sqrt{-g} a^i, \quad X' \equiv X'(\varphi, \psi; \mathbf{f}^2) \equiv \frac{\partial X}{\partial \mathbf{f}^2}, \quad (20)$$

which is useful to write in the LC coordinates  $(u, v)$ . Using the definitions and relations

$$ds^2 = -4h(u, v) du dv, \quad \sqrt{-g} = 2h, \quad f_{uv} \equiv a_{u,v} - a_{v,u}, \quad -2\mathbf{f}^2 = (f_{uv}/h)^2, \quad (21)$$

we easily rewrite equations (20) as

$$\partial_u (h^{-1} X' f_{uv}) = -Z(\varphi) a_u, \quad \partial_v (h^{-1} X' f_{uv}) = Z(\varphi) a_v. \quad (22)$$

Defining now the scalar fields  $q(u, v)$  in the LC or in general coordinates by

$$q(u, v) \equiv h^{-1} X' f_{uv}, \quad 2\sqrt{-g} f^{ij} X' \equiv \varepsilon^{ij} q, \quad i, j = 0, 1. \quad (23)$$

we see that they are constant in the massless case, when  $Z = 0$ , but in general satisfy certain equations which we will derive in a moment. Then, Eqs.(22) allow us to find the vector fields  $\mathbf{f}^2$  once we know the scalar fields  $q$ :

$$a_u(u, v) = -Z^{-1}(\varphi) \partial_u q(u, v), \quad a_v(u, v) = Z^{-1}(\varphi) \partial_v q(u, v). \quad (24)$$

Equations (23) can be rewritten as equations for  $\mathbf{f}^2$  and can in principle be solved. Denoting the solution by  $\bar{\mathbf{f}}^2$  (it depends on  $\varphi, \psi, q$ ), we write them in the form

$$2\bar{\mathbf{f}}^2 = -(q/\bar{X}')^2, \quad \bar{X}' \equiv \frac{\partial}{\partial \bar{\mathbf{f}}^2} X(\varphi, \psi; \bar{\mathbf{f}}^2). \quad (25)$$

Using this solution we can find different expressions for the **effective action** that allows one to derive the explicit equations of motion for the fields  $h, \varphi, \psi, q$ . In [35] we obtained two equivalent expressions for  $X_{\text{eff}}$ <sup>11</sup>

$$X_{\text{eff}}(\varphi, \psi; q) = X(\varphi, \psi; \bar{\mathbf{f}}^2) - 2\bar{\mathbf{f}}^2 \bar{X}' = X(\varphi, \psi; \bar{\mathbf{f}}^2) + q^2/\bar{X}'. \quad (26)$$

From (25) and (26) we also derived in [35] the most compact and beautiful form for  $X_{\text{eff}}$ :

$$X_{\text{eff}}(\varphi, \psi; q) = \left[ X(\varphi, \psi; \bar{\mathbf{f}}^2) + q(u, v) \sqrt{-2\bar{\mathbf{f}}^2} \right]. \quad (27)$$

It is a bit inconvenient because we have to correctly choose the signs of  $q$  and  $\sqrt{-2\bar{\mathbf{f}}^2}$  but in practice it is not difficult. The main difficulty in practical calculations is to explicitly derive  $\bar{\mathbf{f}}^2$  as a function of  $\varphi, \psi, q$  (though for  $D = 3, 4$  it is very easy).

Now we can construct the effective Lagrangian giving all the equations of the new picture. Equations (20) for  $a_u, a_v$  obviously follow from (22), (23) that define the transformation. Calculating  $f_{uv}$  by taking the partial derivatives of  $a_u$  and  $a_v$  in Eq.(24) we find the equations of motion for the scalaron  $q(u, v)$  in the standard form (see Eq.(38) in next Section) but with somewhat unusual function  $\bar{Z}(\varphi) = Z^{-1}(\varphi)$  as can be seen from (24).

As in the case of a massless vector field we can simply replace  $X$  by  $X_{\text{eff}}$ . For the proof of the on-mass-shell equivalence of the effective theory (19) supplemented by definitions and equations (20) - (25) and with  $X_{\text{eff}}$  given by (26) or (27) one can use the following easily checked identities:

$$\frac{dX_{\text{eff}}}{dh} = 0, \quad \frac{dX_{\text{eff}}}{d\varphi} = \partial_\varphi X(\varphi, \psi; \bar{\mathbf{f}}^2), \quad \frac{dX_{\text{eff}}}{d\psi} = \partial_\psi X(\varphi, \psi; \bar{\mathbf{f}}^2). \quad (28)$$

Here we suppose that  $q$  are independent variables and  $\bar{\mathbf{f}}^2$  are the functions of  $\varphi, \psi, q$  satisfying equations (25). To prove these relations we formally differentiate expression (27) and use (25). For example,

$$\frac{dX_{\text{eff}}}{d\varphi} = \partial_\varphi X + \frac{d\bar{\mathbf{f}}^2}{d\varphi} \left[ \bar{X}' - q/\sqrt{-2\bar{\mathbf{f}}^2} \right] = \partial_\varphi X(\varphi, \psi; \bar{\mathbf{f}}^2), \quad (29)$$

and the same vanishing term in brackets emerges in the expression for the second and third terms of (28). The first of identities in (28) is a characteristic property of  $X_{\text{eff}}$  and thus can serve to define it (see [35]).

Thus the solution of the DG coupled to scalar and Abelian gauge fields is reduced to solving DG coupled only to scalars  $\psi$ . The special case, when  $X(\varphi, \psi; \bar{\mathbf{f}}^2) = X(\varphi)\bar{\mathbf{f}}^2$  was known for long time and was used, for example, in finding charged spherical BH solution. The general theorem can be further generalized and applied to much more difficult problems. It was stated in [35] that *"It is not difficult to apply this construction to known Lagrangians of the Dirac - Born - Infeld type as well as to find new integrable models with nonlinear coupling of Abelian gauge fields to gravity."* Now we realize this proposal by further generalizing this theorem that allows us to make a transformation of the neutral massive vector fields into neutral (and effectively massive) scalar fields with a somewhat unusual coupling to gravity.

As a matter of fact, in the above construction we did not use the masslessness of  $a$ . And thus can try to define the effective potential  $X_{\text{eff}}$  for theory (18) by the same equations and definitions (20) - (27). Then, we consider equations (22) as the expression of the vector field in terms of the new scalar field  $q(u, v)$ .<sup>12</sup> It is clear that the vector mass term gives in the transformed Lagrangian

<sup>11</sup>At this point, it is useful to recall that in the massless case  $X = x(\varphi)\bar{\mathbf{f}}^2$  and thus  $X_{\text{eff}} = -x(\varphi)\bar{\mathbf{f}}^2$ , with  $\bar{\mathbf{f}}^2 = q^2/2x^2(\varphi)$ , where  $q$  is a constant charge.

<sup>12</sup>From now on we leave only one vector field and thus omit the subscript.

the kinetic term of the scalaron, while  $X_{\text{eff}}$  defines the nonlinear coupling of the scalaron to DG. More generally, we expect that the two-dimensional DVG (20) can be transformed into DSG

$$\mathcal{L}_{\text{eff}}^{(2)} = \sqrt{-g} \left[ \varphi R + V(\varphi, \psi) + X_{\text{eff}}(\varphi; q(u, v)) + \bar{Z}(\varphi)(\nabla q)^2 + \sum Z(\varphi, \psi)(\nabla \psi)^2 \right], \quad (30)$$

which corresponds to a simplified version with one vector field generating one scalaron  $q(u, v)$ . For the spherically reduced vecton Lagrangian (7) with  $D = 4$  we can easily derive,

$$X_{\text{eff}}(\varphi; q(u, v)) = -2\Lambda\sqrt{\varphi} \left[ 1 + q^2/\lambda^2\Lambda^2\varphi^2 \right]^{\frac{1}{2}}, \quad V = 2k\varphi^{-\frac{1}{2}}, \quad \bar{Z} = -1/m^2\varphi, \quad (31)$$

and there are no  $\psi$ -field terms. For the general cylindrical reduction we may have two  $\psi$ -terms and  $\psi$ -dependent potential  $V$  (see (12)). For the  $D = 3$  case (8) we have

$$X_{\text{eff}}(\varphi; q(u, v)) = -2\Lambda\varphi - q^2/\lambda^2\Lambda\varphi, \quad V = 0, \quad \bar{Z} = -1/m^2\varphi, \quad (32)$$

Now, if we could solve the equations of motion for Lagrangian (30) we would find the vecton field using the simple relation

$$a_u = \partial_u q(u, v)/m^2\varphi, \quad a_v = -\partial_v q(u, v)/m^2\varphi. \quad (33)$$

We do not write here the general equations for DSG theory, the reader can find them in [21], [31]. Most probably, the two-dimensional equations for  $m^2 \neq 0$  are not integrable. Even their one-dimensional reductions, described by ordinary differential equations seem to be not integrable. Below, we employ for their study some convergent or asymptotic expansions. To succeed with this one has to carefully study their general analytic properties and possible singularities.

By the way, the ‘duality’ between vecton and scalaron shows why a perturbation theory with the vecton mass as the parameter must be in some sense singular. In this paper we will not discuss it. The main subject of our study will be the two-dimensional DVG theory

$$\mathcal{L}^{(2)} = \sqrt{-g} \left[ \varphi R + V(\varphi) + X(\varphi; \bar{\mathbf{f}}^2) + Z_a(\phi) \mathbf{a}^2 \right], \quad (34)$$

its dual DSG theory with  $X_{\text{eff}}(\varphi; q(u, v))$  given by (31), (32), and their reductions to dimension 0+1 (static states). The most interesting solutions are those having local horizons which we roughly describe and classify in next Section. We first consider general properties of horizons and conditions for their existence for the general DSG theory given by (30) with fairly arbitrary potentials. Then we discuss more concrete models and solutions with horizons. Our consideration could easily be adapted to cosmological models but we postpone this to near future.

## 4 Horizons in DSG: a general theory

### 4.1 Equations

Consider a general two-dimensional DSG model that embraces all the above Lagrangians as well as some more general not yet discussed:

$$\mathcal{L}_{\text{dsg}} = \sqrt{-g} \left[ \varphi R + U(\varphi, \psi, q) + \bar{Z}(\varphi)(\nabla q)^2 + \sum Z(\varphi, \psi)(\nabla \psi)^2 \right]. \quad (35)$$

To simplify formulas, in the following we consider only one field  $\psi$  and use the LC coordinates. It is not difficult to recover coordinate independent equations with any number of scalar fields (even with the ‘potentials’  $Z(\varphi, \psi)$ ) depending on several fields  $\psi$ . Let us write the equations of motion

in the LC coordinates (for more details see [31],[21]). The energy and momentum constraints should be derived in general coordinates and then rewritten in the LC ones:

$$h \partial_i (\partial_i \varphi / h) = \bar{Z}(\varphi) (\partial_i q)^2 + Z(\varphi, \psi) (\partial_i \psi)^2, \quad i = u, v. \quad (36)$$

The equations of motion can be derived from the LC transformed Lagrangian,

$$\frac{1}{2} \mathcal{L}_{\text{dsg}} = \varphi \partial_u \partial_v \ln |h| + h U(\varphi, \psi, q) - Z(\varphi, \psi) \partial_u \psi \partial_v \psi - \bar{Z}(\varphi) \partial_u q \partial_v q, \quad (37)$$

simply by variations in  $\varphi, \psi, q$ :

$$\partial_u \partial_v \varphi + h U(\varphi, \psi, q) = 0, \quad \partial_u (Z \partial_v q) + \partial_v (Z \partial_u q) + h \partial_q U(\varphi, \psi, q) = 0, \quad (38)$$

$$\partial_u (Z \partial_v \psi) + \partial_v (Z \partial_u \psi) + h \partial_\psi U(\varphi, \psi, q) = \partial_\psi Z(\varphi, \psi) \partial_u \psi \partial_v \psi, \quad (39)$$

We omit the equation derived by variations in  $\varphi$  because it is satisfied as soon as equations (36) and (38) - (39) are satisfied.<sup>13</sup> These equations can be reduced to dimensions 0+1 (static) or 1+0 (cosmological); the model can also describe some gravitational waves (see [21], [37]).

In this paper we only consider static reductions, when all the fields are supposed to depend on one variable  $u + v = \tau$ .<sup>14</sup> The most interesting solutions are those with horizons. We say that a solution has a horizon if the static metric  $h(\tau)$  regarded as a function of  $\varphi$  has a zero at a finite value of  $\varphi$ , i.e.  $h \rightarrow 0$  for  $\varphi \rightarrow \varphi_0$ . Considering the Schwarzschild and Reissner - Nordström (S-RN) horizons<sup>15</sup> one can see that it is more convenient to replace the variable  $\tau$  by  $\varphi$  because then there will be no singularity at the horizon  $\varphi = \varphi_0$ . For DSG theory (35), it was shown in [31], [35], [38] that we can rewrite the equations of motion for some new functions, which are finite and nonvanishing for  $\varphi \rightarrow \varphi_0$ . Moreover, these functions can be expanded in series of powers  $\tilde{\varphi} \equiv \varphi - \varphi_0$  that converge in a neighbourhood of zero (provided that the potentials are analytic in this neighbourhood). This approach is applicable to a most general DG coupled to scalar fields but here we consider only the model (35) with the scalaron and one extra scalar.

Now, supposing that all the fields depend on  $\tau$  and denoting the derivative in  $\tau$  by the dot, we write the dynamical system corresponding to equations (36), (38) - (39) in the form

$$\dot{\varphi} \dot{h}/h + h U + \bar{Z} \dot{q}^2 + Z \dot{\psi}^2 = 0; \quad \dot{\chi} + h U = 0, \quad \dot{\varphi} = \chi, \quad (40)$$

$$\bar{Z} \dot{q} = p, \quad 2\dot{p} + h U_q = 0; \quad Z \dot{\psi} = \eta, \quad 2\dot{\eta} + h U_\psi = Z_\psi \dot{\psi}^2, \quad (41)$$

where  $U_\psi \equiv \partial_\psi U$ ,  $Z_q \equiv \partial_q Z$ , etc. (in our model  $\bar{Z}_q = 0$  and in what follows we take  $Z_\psi = 0$ ). This first-order system is easy to rewrite in a Hamiltonian form, with the first equation in (40) serving as the energy (Hamiltonian) constraint (see [30], [31]). However, representation (40) - (41) (similar to what we used in [31], [35], [38] but with the independent variable  $\varphi$  instead of  $\chi$ ) is more convenient for search and study of horizons. One of the reasons is the following. it is not difficult to show that  $h \neq 0$  for finite  $\tau$ , and thus to find solutions near horizons we should work in asymptotic regions with, probably, asymptotic expansions. In addition, the distinction between regular and singular horizons discussed below is, at least, not evident.

<sup>13</sup>It simply gives the expression of the scalar two-dimensional curvature  $R \equiv (\partial_u \partial_v \ln |h|)/h$  in terms of the  $\varphi$ -derivative of the other terms in the Lagrangian. It may be useful in search for additional integrals, e.g., [31].

<sup>14</sup>We here use the notation  $\tau$  instead of  $r$  because  $\tau$  does not coincide with the ‘radius’ in the static coordinates. We will see that a horizon can emerge when  $\tau \rightarrow \infty$ . In our picture the role of a ‘radius’ plays the dilaton  $\varphi$  which is finite at the horizon. The distinction between the space variable  $r$  and time variable  $t$  can be established when we return to a higher dimensional origin of DSG. Note also that, due to the residual coordinate invariance, we can equivalently use transformed LC coordinates  $a(u), b(v)$  in which  $ds^2 = -4h a'(u) b'(v)$ .

<sup>15</sup>One can find a clear and concise LC treatment of black holes in [26].

## 4.2 Examples of horizons

Let us illustrate derivations of horizons by considering simplest system (35) with  $U = U(\varphi)$ ,  $q = q_0 = \text{const}$  (massless vecton), and no extra scalars. Using (40), with  $q = q_0$ ,  $\psi \equiv 0$ , we find

$$h = C_0 \dot{\varphi} \mapsto \ddot{\varphi} + C_0 \dot{\varphi} U(\varphi) = 0 \mapsto \dot{\varphi} + C_0 N(\varphi) = C_1,$$

where  $U(\varphi) = U(\varphi, 0, q_0)$ ,  $N(\varphi) \equiv \int U(\varphi) d\varphi$ ,  $C_0, C_1$  are constants. It follows

$$h = C_0^2 [N_0 - N(\varphi)], \quad C_0 \tau = \int d\varphi [N_0 - N(\varphi)]^{-1}, \quad (42)$$

where  $N_0 = C_1/C_0$  defines a finite position of the horizon in the interval  $0 < \varphi_0 < \infty$  by the equation  $N(\varphi_0) = N_0$ . As  $N(\varphi)$  is continuous (and differentiable if  $U(\varphi)$  is continuous), there exists at least one horizon for a generic potential. In the special cases of the Schwarzschild - Reissner-Nordstroem (S-RN) horizons Eq.(42) gives the complete standard description of the black holes in any space-time dimension.<sup>16</sup> The solution (42) is valid for any potential  $U(\varphi)$  and describes a more general objects with any number of horizons that may be more complicated than than in the S-RN case. The S-RN black holes are the simplest examples of **regular** and **simple** (non - degenerate) horizons. In addition, the RN horizons may be double **degenerate**, the corresponding black holes are then called extremal black holes. The potential describing these as well as somewhat more general black holes (with the  $\Lambda$  term) can be derived from the formulas of Section 3,

$$U(\varphi) = k_\nu \varphi^{-\nu} - 2\Lambda \varphi^\nu - q_0^2 \varphi^{\nu-2}, \quad \nu \equiv 1/(D-2). \quad (43)$$

It is not difficult to rewrite the S-RN solution (42) in the Schwarzschild, Eddington - Finkelstein or Szekeres - Kruskal (SK) coordinates. We will show how to construct a local analog of the solution (42) for a general non-integrable theory (35), demonstrate that all the mentioned types of horizons exist generally, and, by the way, introduce a local generalization of the SK coordinates.

Before turning to this construction we discuss some general properties of the horizons described by Eq.(42). If  $N'(\varphi_0) = 0$ , the horizon is (double) **degenerate**, i.e., it can be obtained by a fusion of two simple (nondegenerate) horizons. The additional condition for triple degeneracy of  $\varphi_0$  is  $U'(\varphi_0) = 0$ . This is possible if there are two relations between parameters determining the potential  $U$ , for (43) - between  $\Lambda$ ,  $q_0$ ,  $k_\nu$ . Therefore, in principle, there may exist in this case a triple degenerate horizon. If one of the parameters vanishes, only the double degeneracy is possible: 1.  $\Lambda = 0$ ,  $k_\nu > 0$ ; 2.  $\Lambda < 0$ ,  $k_\nu = 0$ ; 3.  $\Lambda k_\nu > 0$  for  $q_0 = 0$ . It is not difficult to find that both conditions for the triple degenerate horizon can be satisfied if  $\Lambda < 0$ ,  $k_\nu > 0$  and

$$q_0^2 (2\Lambda)^{D-3} = (D-3)^{D-3} (\nu k_\nu)^{(D-2)} \equiv (D-3)^{2D-5},$$

where we used the definition of  $k_\nu$ .

The maximal number of possible horizons for an arbitrary potential  $N(\varphi)$  can be derived by finding the maximally degenerate horizon. The necessary conditions for this simple derivation are: the potential must be sufficiently differentiable and continuously depending on the parameters, which can be deformed to make all the horizons to merge. Then the **number of possible horizons** coincides with the **maximal degeneracy** of them. A local generalization of this ‘theorem’ looks not so useful, being, at first sight, just a simple statement about splitting a degenerate horizon into simple ones. However, some global information could probably be extracted from more easily accessible local structure of horizons. We cannot say more at the moment but it would be useful to analyze this problem on different interesting examples.

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<sup>16</sup>See, e.g., [26], [31] and references therein. To get the standard formulas for the metric we must recall about the Weyl transformation of the metric (we use transformed Lagrangians (7) instead of the original ones (4)), return to the space-time coordinates and take into account that  $\varphi = r^{D-2}$ .

A very simple integrable model with a **singular** horizon was given in [38]). This model can be obtained from Eq.(40) - (41) by recalling that the first equation in (40) is the energy condition stating that the Hamiltonian must vanish. Thus we can write the Hamiltonian

$$\mathcal{H} = \dot{\varphi} \dot{h}/h + hU + \bar{Z} \dot{q}^2 + Z \dot{\psi}^2 \quad (44)$$

introduce the canonical variables, and derive the canonical equations, which essentially coincide with (40)-(41) except for one extra equation giving  $(\ln h)''$  in terms of the other variables (we omitted this equation above as it directly follows from the other ones). Now, suppose that  $\bar{Z} = -1$ ,  $Z = 0$  and  $U = gq^n$ .<sup>17</sup> As  $\mathcal{H}$  is independent of  $\varphi$ , we have  $h = \exp(a + b\tau)$  and it is easy to find that our equations are integrable for  $n = 1$  (trivial) and for  $n = 2$  (not quite trivial but simple enough). In the  $n = 1$  case, the main result is

$$q = q_0 + h(g/2b^2), \quad \tilde{\varphi} = q_0 h(g/b^2) - h^2(g^2/8b^4). \quad (45)$$

This equations define a regular simple horizon if  $q_0 \neq 0$ , but when  $q_0 = 0$  we find  $h \sim \pm \sqrt{|\tilde{\varphi}|}$ .

This example is important because it shows that singular horizons can be produced by completely regular potentials. Of course, if the potentials are not regular we expect that correspondingly there may exist singular horizons. To illustrate this by a simple and sufficiently realistic example we return for a moment to the pure DG theory the general solution of which is given by Eq.(42). If  $N(\varphi)$  is analytic for  $\varphi > 0$ , all horizons are regular. The points  $\varphi_0$  where  $N$  is nonanalytic may give a singular horizons. Consider the DG potential given by (32), in which the  $q$ -field is a constant  $q_0$  and take  $\lambda^2 < 0$ .<sup>18</sup> Neglecting the multipliers  $\sim \varphi^{-\frac{1}{2}}$  that are analytic on  $\varphi > 0$  we take the approximation preserving the singularity of the potential,

$$U(\varphi) = -2\Lambda\sqrt{\varphi^2 - q_r^2} \equiv -2\Lambda\sqrt{\varphi_0^2 - q_r^2 + 2\varphi_0\tilde{\varphi} + \tilde{\varphi}^2},$$

where  $\varphi \equiv \varphi_0 + \tilde{\varphi}$  and  $q_r$  is the renormalized  $q_0$ . Thus, for  $\varphi_0^2 = q_r^2$  the potential is proportional to  $\sqrt{\tilde{\varphi}}$  and  $N(\varphi)$ ,  $h(\varphi)$  have the singularity  $\sim \tilde{\varphi}^{3/2}$ .

We believe that singular horizons must be regularly met in gravity but at the moment there are not many realistic examples. Probably, the reason is that all regular horizons (including the degenerate ones!) are similar in their construction but singular ones are singular each in their own way.

### 4.3 Solutions near horizons

Now we use as the independent variable  $\varphi$  instead of  $\tau$ . Denoting the differentiation in  $\varphi$  by the prime and defining  $h/\chi \equiv H$ ,  $p/\chi \equiv P$ ,  $\eta/\chi \equiv E$ , we can transform the system (41) into the following equivalent dynamical system ( $Z_q = Z_\psi = 0$ )<sup>19</sup>:

$$\begin{aligned} q' &= P\bar{Z}^{-1}, \quad \psi' = EZ^{-1}, \quad (\chi P)' = -\frac{1}{2}HU_q, \quad (\chi E)' = -\frac{1}{2}HU_\psi, \\ \chi' &= -HU, \quad H' = -H(P^2\bar{Z}^{-1} + E^2Z^{-1}). \end{aligned} \quad (46)$$

Suppose that  $q(\varphi)$ ,  $\psi(\varphi)$ ,  $U(\varphi, q, \psi)$ ,  $\bar{Z}^{-1}(\varphi)$ , and  $Z^{-1}(\varphi)$  are finite when  $\tilde{\varphi} \rightarrow 0$  and can be expanded in power series in  $\tilde{\varphi}$ , while  $h/\tilde{\varphi} \rightarrow h_1 \neq 0, \infty$ . It then follows that  $H, E, P$  must be

<sup>17</sup>Hamiltonians of this sort can be met in simple approximations for branes (see, e.g. Cavaglia - Gregory). As we work near the horizon  $\varphi_0$  we can approximate the dilaton field  $\varphi$  by  $\varphi_0$  in  $\bar{Z}$  and in  $U$ .

<sup>18</sup>This is not unphysical assumptions for  $\Lambda < 0$  because it corresponds to the massless vector theory (4) - (5) with the correct sign of the kinetic term  $-\Lambda\lambda^2\varphi\mathbf{f}^2$  for small field values.

<sup>19</sup>This is not important for our argument but simplifies the expansions below. In a moment we further simplify the equations by neglecting the  $\psi$ -field which may emerge in cylindrical reductions.

finite in this limit and can be expanded in power series. In particular, we find that  $H \rightarrow H_0$  and from equations (46) we have  $\chi \rightarrow 0$ ,  $P \rightarrow P_0$ ,  $E \rightarrow E_0$ , i.e.  $p \rightarrow 0$ ,  $\eta \rightarrow 0$ . We thus can write

$$\begin{aligned} h &= \sum h_n \tilde{\varphi}^n, & \chi &= \sum \chi_n \tilde{\varphi}^n, & q &= \sum q_n \tilde{\varphi}^n, & \psi &= \sum \psi_n \tilde{\varphi}^n, \\ H &= \sum H_n \tilde{\varphi}^n, & P &= \sum P_n \tilde{\varphi}^n, & E &= \sum E_n \tilde{\varphi}^n, \end{aligned} \quad (47)$$

where summation is over  $0 \leq n < \infty$  and, of course,  $h_0 = \chi_0 = 0$ . Using these expansions we can expand the potentials in powers of  $\tilde{\varphi}$ . To simplify notation we neglect the extra scalar and take  $\psi = E = U_\psi = 0$ . The expansion of  $U$ ,  $U_q$ ,  $\bar{Z}^{-1}$  can be explicitly derived by the expansions of  $\varphi$ ,  $q$  from (47) (in our models (30) - (32) we have simply  $\bar{Z}^{-1} = -m^2 \varphi \equiv -m^2(\varphi_0 + \tilde{\varphi})$ ). To introduce notation we write a generic expansion of  $X(\varphi, q)$  and of  $X(\varphi, q)Y(\varphi, q)$ :

$$\begin{aligned} X(\varphi, q) &= \sum X^{(n)} \tilde{\varphi}^n, & X^{(0)} &= X(\varphi_0, q_0), & X^{(1)} &= X_\varphi^{(0)} + q_1 X_q^{(0)}, \\ X(\varphi, q)Y(\varphi, q) &= \sum (XY)^{(n)} \tilde{\varphi}^n, & (XY)^{(n)} &\equiv \sum_{m=0}^n (X)^{(n-m)}(Y)^{(m)}, \end{aligned} \quad (48)$$

where  $X, Y$  may, in particular, coincide with one of the functions in (46).

With the assumptions formulated above, we can write the recurrence relations determining the coefficients of the power series (47):

$$\begin{aligned} (n+1)\chi_{n+1} &= -(UH)^{(n)}, & 2(n+1)(\chi P)^{(n+1)} &= -(U_q H)^{(n)}, \\ (n+1)q_{n+1} &= (\bar{Z}^{-1}P)^{(n)}, & (n+1)H_{n+1} &= -(\bar{Z}^{-1}P^2 H)^{(n)}. \end{aligned} \quad (49)$$

The ‘initial’ values  $q_0, H_0$  and the position of the horizon,  $\varphi_0$ , are arbitrary. Then we find

$$\chi_1 = -U^{(0)}H_0, \quad P_0 = U_q^{(0)}/2U^{(0)}, \quad q_1 = \bar{Z}_{(0)}^{-1}P_0, \quad H_1 = -q_1 P_0^2 H_0. \quad (50)$$

Having this quadruple we can recursively derive any quadruple  $\chi_{n+1}, P_n, q_{n+1}, H_{n+1}$  in terms of  $\varphi_0, q_0, H_0$ . It is not difficult to find several first quadruples although we do not know a general expression for any  $n$ . For further discussions we need  $\chi_2, h_2$  ( $h_1 = H_0 \chi_1$ ):

$$\begin{aligned} \chi_2 &= -\frac{1}{2}(U^{(0)}H_1 + U^{(1)}H_0) = -\frac{1}{2}(U_\varphi^{(0)} + U^{(0)}P_0^2 \bar{Z}_{(0)}^{-1})H_0, \\ h_2 &= H_0 \chi_2 + H_1 \chi_1 = -\frac{1}{2}(U_\varphi^{(0)} - U^{(0)}P_0^2 \bar{Z}_{(0)}^{-1})H_0^2. \end{aligned} \quad (51)$$

Expansions (47) exist if the potentials do not vanish at  $\varphi = \varphi_0$ , i.e.,

$$U^{(0)} \equiv U_{(0)} \equiv U(\varphi_0, q_0) \neq 0, \quad \bar{Z}^{(0)} \equiv \bar{Z}_{(0)} \equiv \bar{Z}(\varphi_0, q_0) \neq 0. \quad (52)$$

For DSG theory (35) (with  $Z \equiv 0$ ), which is dual to DVG theory (34) and has the effective potentials  $U = X_{\text{eff}} + V$  and  $\bar{Z}$  given by Eqs.(31) - (32), these conditions are satisfied if  $\Lambda \neq 0$  and  $0 < \varphi < \infty$ . Then the power series solutions are well defined and converge near  $\tilde{\varphi} = 0$  as argued in [38].<sup>20</sup> As we have seen above, there exist various obstructions to convergence: there can be singularities in the potentials similar to the discussed above; in addition, our nonlinear, non-integrable system can produce so called ‘moving singularities’ depending on the initial values and parameters (like the critical dependence on  $q_0$  in (45)).<sup>21</sup> A natural approach to singularities might be to find the leading singularity and try to look for corrections in the form of a convergent or asymptotic power series multiplier, like  $\sqrt{\tilde{\varphi}} \sum a_n \tilde{\varphi}^n$ . We will not pursue this idea further.

<sup>20</sup>If the first of conditions (52) is not satisfied then to have a finite limit for  $P_0$  the derivative  $U_q^{(0)}$  must also vanish (see (50)). Then we see that  $\chi_1 = h_1 = 0$  but  $\chi_2$  and  $h_2$  do not vanish. This is a characteristic property of (double) degenerate horizons. It is important that our expansion works also in this degenerate case. A general treatment of degenerate horizons will be published elsewhere.

<sup>21</sup>Here we should emphasize a peculiarity of our problem (46). While we have four unknown functions (with  $\psi = E = U_\psi = 0$ ) and, at a first sight, a standard Cauchy problem  $\chi(0) = 0$ ,  $H(0) = H_0$ ,  $q(0) = q_0$ ,  $P(0) = P_0$ , in reality  $P_0$  is expressed by (4.13) in terms of  $q_0$  and  $\varphi_0$  and thus cannot be arbitrary chosen (in our setting  $\varphi_0$  is an arbitrary parameter of the system, not the initial condition).

#### 4.4 Generalization of Szekeres - Kruskal coordinates

In conclusion of this Section we introduce a generalization of the SK coordinates for any simple regular horizon for which the metric in (21) is  $h = h_1\tilde{\varphi} + h_2\tilde{\varphi}^2 + \dots$ , with  $h_n$  given by (49) - (52). As the metric changes sign with  $\tilde{\varphi}$ , from the physics point of view there is a ‘transition’ of the static geometry to a time-dependent one. In the Schwarzschild coordinates, the metric is infinite at the horizon and also changes sign with  $r - r_0$ . This ‘singularity’ disappears if one uses SK coordinates, the analog of which is easy to define by a simple change of the chiral (LC) coordinates  $u \mapsto a(u)$ ,  $v \mapsto b(v)$ :

$$u = \frac{\ln a(u)}{\chi_1}, \quad v = \frac{\ln b(v)}{\chi_1}; \quad ab = \exp[\chi_1(u + v)] \equiv \exp(\chi_1\tau) = \exp \int d\varphi \frac{\chi_1}{\chi(\varphi)}. \quad (53)$$

In view of the above expansions the last expression behaves for  $\tilde{\varphi} \rightarrow 0$  like

$$\exp \int d\varphi \frac{\chi_1}{\chi(\varphi)} = \tilde{\varphi} \exp \left[ -\frac{\chi_2}{\chi_1} \tilde{\varphi} + O(\tilde{\varphi}^2) \right], \quad (54)$$

where  $\tilde{\varphi} \equiv \varphi - \varphi_0$  and  $O(\tilde{\varphi}^2)$  is a locally convergent power series. This means that the metric has no zero at  $\tilde{\varphi} \rightarrow 0$ . Using (50) - (54) we rewrite the metric in  $(a, b)$  coordinates:

$$ds^2 = -4h(u + v) du dv \equiv -4h_{\text{sk}}(ab) da db = -4 \frac{h(\varphi)}{\chi_1^2 ab} da db \rightarrow -4 \frac{da db}{U^{(0)}}. \quad (55)$$

It is not difficult to derive first terms of the expansion of  $h_{\text{sk}}(\tilde{\varphi})$ :

$$h_{\text{sk}}(\tilde{\varphi}) = \frac{h_1}{\chi_1^2} \left[ 1 + \left( \frac{h_2}{h_1} + \frac{\chi_2}{\chi_1} \right) \tilde{\varphi} + \dots \right] = -\frac{1}{U^{(0)}} \left[ 1 + \frac{U_\varphi^{(0)}}{U^{(0)}} \tilde{\varphi} + \dots \right]. \quad (56)$$

Interesting enough, these first terms depend only on  $q_0$  in  $U^{(0)}$  and  $U_\varphi^{(0)}$ , not on  $U_q^{(0)}$ , which is proportional to  $q_1$ .<sup>22</sup> Let us compare this result with the SK transformation for S-RN black holes. In this case we can write the closed expression for the transformed metric (see (42) with  $C_0 = 1$ ):

$$h_{\text{sk}}(\varphi) = \frac{[N_0 - N(\varphi)]}{U_0^2} \exp \left( -U_0 \int d\varphi [N_0 - N(\varphi)]^{-1} \right), \quad (57)$$

where  $U_0 \equiv U(\varphi_0)$ ,  $N'(\varphi) \equiv U(\varphi)$ , etc. It is easy to find that the first approximation to this exact result coincides with Eq.(56) (of course, the higher terms essentially depend on  $q$ ).

To compare our coordinates with SK ones, we briefly show how to rewrite our LC results in the standard Schwarzschild coordinates. First rewrite the metric in terms of coordinates  $\tau = u + v \equiv r$  and  $t = u - v$ , so that  $4dudv = dr^2 - dt^2$  and thus

$$ds^2 = -4h(\varphi) du dv = -h(\varphi) \left[ \frac{d\varphi^2}{\chi^2(\varphi)} - dt^2 \right] \equiv -H(\varphi) \left[ \frac{d\varphi^2}{\chi(\varphi)} - \chi(\varphi) dt^2 \right]. \quad (58)$$

To get the standard form of the static metric we must first to make the inverse Weyl transformation, i.e. to divide this metric by  $w(\varphi) = \varphi^{1-\nu}$  (it is the same for the spherical and cylindrical reductions, see (6)) and to replace  $\varphi = r^{D-2}$ ,  $w = r^{D-3}$ . Then we have the desired solution near a horizon in  $(r, t)$  coordinates:

$$ds^2 = -(D-2) H_s(r) \left[ \frac{dr^2}{\chi_s(r)} - \chi_s(r) dt^2 \right]; \quad \chi_s(r) \equiv \nu \chi(r^{D-2}) r^{3-D}, \quad (59)$$

<sup>22</sup>The first two terms in the expansion are insensitive to the parameters of the scalar fields and the expression for them coincide with the DG one.

where  $H_s(r) \equiv H(r^{D-2})$ . Reproducing the standard S-RN black hole solutions is now trivial:  $H \equiv H_0 = C_0$  and  $\chi(\varphi)$  is given by Eq.(42); deriving  $N(\varphi)$  for the potential (43) and adjusting notation we get the general spherical black hole metric in all the discussed coordinate: LC, Schwarzschild, and SK. Our most general solutions are only valid near horizons but, if we succeed in analytic continuation of them up to infinity  $r^{D-2} = \varphi \rightarrow \infty$  or to singularity at the origin  $\varphi = 0 = r$ , this solution may be regarded as global as S-RN ones. Hopefully, it may be possible for some solutions with maximally degenerate horizons, at least for special values of the parameters

## 5 Quasi summary and short remarks

The main results of this papers are the following. In Section 3 we proposed the transformation of the DVG into the equivalent DSG. This allows applying to the vecton models some methods developed in two-dimensional dilaton gravity models with scalar fields. In particular, the nonlinear kinetic terms of the vecton theory transform into completely standard potentials depending only on scalar fields (dilaton, scalaron, other scalars). The scalar form makes it easier to look for additional integrals of motion in the one-dimensional reductions of DSG (we may call them DSG1).<sup>23</sup> In [31] (pp. 1698-1699) we derived two nontrivial DG models coupled to scalar fields and having two additional integrals. Using these integrals it was possible to integrate them. We only mention here the first model of Ref.[31] that can be generalized to some effectively massive scalar scalar fields. It is not difficult to show that for ‘multiplicative potentials  $V = u(\varphi)v(\psi)$ ’ there exists an additional integral of motion if the kinetic potential  $Z(\varphi)$  is related to  $u(\varphi)$  as

$$Z(\varphi) = (g_0/u(\varphi)) \int u(\varphi) d\varphi, \quad (60)$$

while  $v(\psi)$  is arbitrary function ( $g_0$  is arbitrary constant). The integral is insensitive to  $v(\psi)$  and is the same as given in [31] (for simplicity we choose there the Weyl frame in which  $W = 0$ ):

$$Zh^{-1}\dot{h} - g_1\partial\varphi = C_0, \quad (61)$$

where  $g_1$  is a constant depending on the integration constant in the definition of  $Z$ . This result may be of use in analyzing DSG introduced in Section 3.

The second integrable model in [31] deserves special attention because it allows us to introduce a concept of *the topological portrait* describing qualitative properties of static and cosmological solutions for different values of a parameter  $\delta$  characterizing the energy of the massless scalar field. Introducing the scales  $w_0$  and  $h_0$  for  $w(\varphi)$  and  $h(\varphi)$  we find the relation between normalized  $w$  and  $h$  that depends only on  $\delta$  varying in the interval  $[-\frac{1}{2}, +\infty]$ :

$$w = \frac{|h|^\delta}{|1 + \varepsilon|h|^{1+2\delta}|}. \quad (62)$$

Here  $\varepsilon \equiv h/|h|$ ,  $-1 < h < \infty$ ,  $0 < w < \infty$ . It is not very difficult (but not quite easy) to draw the picture of the curves describing all possible solutions. In the domain  $h < 0$  we have static solutions, while for cosmological ones  $h > 0$ . The picture looks like a phase portrait of a dynamical system in the  $(h, w)$ -plane, with singular points: 0)  $(0, 0)$ , 1)  $(0, 1)$ , 2)  $(-1, \infty)$ , 3)  $(1, \frac{1}{2})$ , 4)  $(0, \infty)$ , 5)  $(\infty, 0)$ , 6)  $(-1, 0)$ . These points are joined by the important separating curves. The most interesting points are: the node of the initial singularity, (p.0), the saddle point of the horizon, (p.1), and the cosmological point (p.3), at which all cosmologies tangentially coincide.

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<sup>23</sup>Pure dilaton gravity is a topological theory and thus reduces to the one-dimensional integrable system. There exist simple enough examples of integrable systems involving one massless scalar field in addition to the dilaton (see [31]). More complex models with effectively massive scalar field may have one additional integral, at best, and thus remain nonintegrable.

While Section 4 presents a complete description of static solutions near horizons, the topological portrait, if available, will presumably allow us to find the global picture and to make more clear the relation between static and cosmological solutions. It would be very helpful to find an approach to drawing 3D-portraits of integrable DSG systems.

## 6 Appendix

In this paper we use different coordinates in two-dimensional and one-dimensional dilaton theories. To make reading easier we give here short comments on the relation between the Lagrangians in the  $(r, t)$  and  $(u, v)$  coordinates. We first write metric (3) in the  $(r, t)$  coordinates,

$$ds_D^2 = e^{2\alpha} dr^2 + e^{2\beta} d\Omega_{D-2}^2 - e^{2\gamma} dt^2 + 2e^{2\delta} dr dt. \quad (63)$$

Here the last term is needed to derive the momentum constraint, and while passing to the one-dimensional coordinates we omit it (formally, by  $\delta \rightarrow -\infty$ ). Then action (4) can be rewritten,

$$\mathcal{L}_D^{(2)} = e^{\alpha+(D-2)\beta+\gamma} \left[ R^{(2)} + \frac{1-\nu}{\nu^2} [e^{-2\beta} + (\nabla\beta)^2] - 2\Lambda [1 + \frac{1}{2}\lambda^2 \mathbf{f}^2]^\nu - m^2 \mathbf{a}^2 \right], \quad (64)$$

where we omit the contribution of the non-diagonal term in metric (63), because we use this expression only to connect one-dimensional reductions with DVG ( $\nu \equiv (D-2)^{-1}$ ). The first two terms in (64) belong to the standard Einstein gravity and the rest is the trace of the affine gravity. In the main text we also used the Lagrangians with additional scalar terms, like (18) (one should keep in mind that its vector part is the Weyl transformed version of Lagrangian (4)).

Considering the expression for the curvature  $R^{(2)}$  in the diagonal metric

$$R^{(2)} = 2[e^{-2\gamma}(\ddot{\alpha} + \dot{\alpha}^2 - \dot{\alpha}\dot{\gamma}) - e^{-2\alpha}(\gamma'' + \gamma'^2 - \gamma'\alpha')], \quad (65)$$

it is easy to find that the  $R^{(2)}$ -term in (64) can be transformed into total derivatives and the terms containing only first-order derivatives. Neglecting the total derivatives we find that the remaining Einstein gravity terms in (64) have the form

$$\mathcal{L}_{DE}^{(2)} \equiv (D-2)(D-3) e^{\alpha+(D-2)\beta+\gamma} \left[ e^{-2\beta} + \beta'^2 - \dot{\beta}^2 + 2(D-3)^{-1}(\beta'\gamma' - \dot{\beta}\dot{\gamma}) \right]. \quad (66)$$

Using this expression it is easy to construct simple dimensional reductions to static or cosmological solutions (or employ more general methods of separation of variables, see [21], [37]). Of course we may apply the same approach to the Weyl transformed theory of DSG and rewrite Lagrangian (35) in metric (63) as follows ( $\varphi = \exp[(D-2)\beta]$ ):

$$\mathcal{L}_{\text{dsg}} = \mathcal{L}_{DE}^{(2)} + e^{\alpha+\gamma} \left[ U(\varphi, \psi, q) + \bar{Z}(\varphi)(q'^2 - \dot{q}^2) + \sum Z(\varphi, \psi)(\psi'^2 - \dot{\psi}^2) \right]. \quad (67)$$

This is a rather general formulation of dilaton gravity coupled to scalars, which is convenient to apply to many problems. A general approach to horizons was presented above. Many integrable cases were studied in literature, most relevant to the present paper are [31], [39], [40], [37].

In Section 4 we derived one-dimensional equations directly from the two-dimensional LC equations (36) - (39). There, we could also dimensionally reduce Lagrangian (37) and obtain the same one-dimensional equations. Alternatively, one may dimensionally reduce Lagrangian (67) and obtain more general equations that allow gauge fixing. For example we can return to the LC gauge by choosing  $\alpha = \gamma$  (this is possible also for the two-dimensional solutions). For static reductions we can choose the Schwarzschild gauge  $\alpha(r) = -\gamma(r)$ . In cosmological reductions a

natural gauge is  $\gamma(t) = 0$ . The LC equations are most suitable for studies of the states on both sides of horizons as well as for deriving solutions near horizons in general non - integrable theories.

To avoid some small but annoying sign problems in relating the  $(u, v)$  and  $(r, t)$  pictures, let us explicitly write the conventions we are using in transitions between the pictures. We always use the following definition of the LC metric (often with  $f(u, v)$  instead of  $h(u, v)$ )

$$ds_2^2 = e^{2\alpha}(dr^2 - dt^2) \equiv -4h(u, v) du dv. \quad (68)$$

Then it is clear that the sign of  $h$  must be positive for the space - like metric (this is true for the exterior space of the Schwarzschild black hole) and negative for the time - like one. For this reason, we choose the relation between the LC and space - time coordinates

$$t = u + \varepsilon v, \quad r = u - \varepsilon v, \quad \varepsilon \equiv |h|/h. \quad (69)$$

With this definitions,  $\partial_t^2 - \partial_r^2 = \varepsilon \partial_u \partial_v$  and the LC curvature is

$$R_{LC}^{(2)} = 2e^{-2\alpha}(\ddot{\alpha} - \alpha'') = h^{-1} \partial_u \partial_v \ln |h|, \quad \sqrt{-g} R_{LC}^{(2)} = 2|h| R_{LC}^{(2)} = 2\varepsilon \partial_u \partial_v \ln |h|. \quad (70)$$

Note that. if we take in all cases  $\sqrt{-g} = 2h$  instead of  $2|h|$ , the equations of motion remain correct though the Lagrangians may change sign. Finally, in one-dimensional theories we usually take as the independent variable  $\tau = u + v$ , which is  $r$  for static states and  $t$  for cosmologies.

We also should mention simple definitions related to the vecton components:

$$a_u = a_0 + a_1, \quad \varepsilon a_v = a_0 - a_1, \quad a_i a^i = -a_u a_v / h, \quad (71)$$

$$f_{01} \equiv \partial_0 a_1 - \partial_1 a_0 = -\frac{1}{2} \varepsilon (a_{u,v} - a_{v,u}) \equiv \frac{1}{2} \varepsilon f_{uv}. \quad (72)$$

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